Analysis and Review of Risk Measures: VAR, CVAR, What to Use?

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Abstract

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Our paper studies the properties of risk measures that can help avoid some inconsistencies observed with popular measures, like VaR or Conditional Value at Risk (CVaR).

Two new families of risk measures are defined; complete and adapted risk measures.

In particular, we study distortion risk measures and characterize them as complete/adapted, in terms of the linearity of the distortion function, rather than its non–differentiability, as suggested by Wang (2002, AFIR Coll.).

Overview:

1. Introduction

2. Properties of risk measures

3. Distortion risk measures
   3.1 Complete risk measures
   3.2 Exhaustive risk measures
   3.3 Beyond exhaustive risk measures

1. Introduction

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Value at Risk (VaR) has been the preferred choice in industry to estimate market risk.

Value at Risk:

VaR can be defined as [Duffie and Pan (1997, J. of Der.)]:

> *For a given time horizon T and a $\alpha \times 100\%$ confidence level, VaR is the loss in market value that can only be exceed with a probability of at most $1 - \alpha$.*

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Mathematically, VaR is simply a percentile on the distribution of losses.

For a risk X over a given period $[0, T]$ and $0 < \alpha < 1$, the $\alpha \times 100\%$ VaR, denoted $VaR_\alpha(X)$, is:

$$VaR_\alpha(X) = \inf\{x \in \mathbb{R} \mid \mathbb{P}(X \leq x) \geq \alpha\}.$$

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- lack of sub–additivity [Artzner et al. (1997, Risk, 1999, Math. Fin.)] and/or of convexity,
- measure difficult to optimize as it may have multiple local minima [Basak and Shapiro (2001, Rev. Fin. St.)].

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With the concept of coherent risk measure and its properties, different sets of measures started to appear, each with distinctive properties: e.g. convex, spectral or deviation measures.

We study distortion risk measures and their properties. Some of these measures do not satisfy all the properties required to avoid inconsistent decisions.

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To avoid the problem, completeness is proposed, to ensure that the measure uses all the information in the loss distribution.

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The inconsistencies are due to the linearity of the distortion rather than its non–differentiability, as suggested in Wang (2002, AFIR Coll.).

Generally completeness is not sufficient to avoid inconsistencies. Adaptability is defined, forcing the risk measure to use the distributional information adequately.

2. Properties of risk measures

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Definition 2.1: A risk measure is a function ρ that associates the index (real) value $\rho(X)$ to a risk X.

Risk Measures in Finance:

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By contrast, if $\rho(X) < 0$ then the amount $-\rho(X)$ can be cashed from the position.

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3. Translation invariance: For a fixed risk $X \in \mathbb{X}$ and any constant a, then $\rho(X+a) = \rho(X) - a$.
4. Monotonicity: Let X, Y be risks such that $X \leq Y$ then $\rho(X) \leq \rho(Y)$.

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Coherent risk measures are characterized in terms of scenarios. In fact, the selection of a risk measure is equivalent to the selection of a set of generalized portfolio scenarios.

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An alternative measure for losses is Conditional Value at Risk (CVaR), also called Tail Conditional Expectation (TCE), sometimes erroneously. It was first proposed by Artzner et al. (1999, Math. Fin.) and extensively studied in recent years.

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CVaR answers the question: What is the expected loss incurred in the $(1-\alpha)\%$ worse cases of the position?

CVaR (. . . continued)

Definition 2.3: For a risk X and a confidence level $0 < \alpha < 1$, the $\alpha \times 100\%$ CVaR is:

$$CVaR_\alpha(X) = \mathbb{E}_P[X \mid X \geq VaR_\alpha(X)].$$

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CVaR is sub–additive and convex. In fact, in most cases, it is coherent [Artzner et al. (1999, Math. Fin.)].

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Convexity: $\rho[(1-\lambda)Y + \lambda X] \leq (1-\lambda)\rho(Y) + \lambda\rho(X)$, for any $\lambda \ \in [0,1]$.

Spectral Risk Measures

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Clearly, opinions differ on the properties that risk measures should satisfy.

Actuarial Risk Measures:

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Desirable properties differ when a risk measure is used for capital requirements, for statutory purposes, compared to risk premium calculations.

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g transforms S_X into a new survival function $S_X^* = g(S_X)$, of the ground–up distribution.

It can be shown that the Choquet integral of a loss X

$$H_g[X] = \int_0^\infty g[S_X(x)]dx - \int_0^\infty \{1 - g[S_X(-x)]\}dx,$$

is equal to the expectation of X under the distorted distribution.

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Wirch and Hardy (2001, IME Conf.) show that distortion risk measures are coherent if and only if g is concave.

In addition, if g is concave then the distortion risk measure is spectral. VaR, CVaR and the WT measure of Wang (2000, J. Risk & Ins.) are all distortion risk measures.

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VaR can be expressed as the Choquet integral with respect to the following distortion function:

$$g(y) = \begin{cases} 0 & \text{if } y < 1-\alpha \\ 1 & \text{if } y \geq 1-\alpha \end{cases}. \tag{1}$$

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$$VaR_\alpha(X) = \int_0^\infty g[S_X(y)]dy - \int_0^\infty \{1 - g[1 - S_X(-y)]\}dy = x_\alpha \ ,$$

where x_α is the percentile of the distribution of X.

Complete (...continued):

Note from the following graph that here g is non–decreasing, with $g(0) = 0$ and $g(1) = 1$, piece–wise constant and non–concave. Hence the resulting distortion risk measure, VaR, is not coherent.

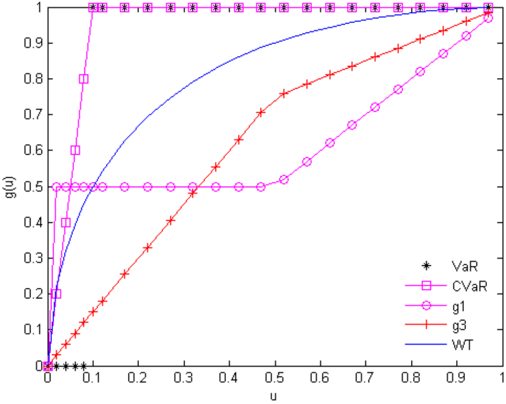

Figure: Distortion functions

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Here g is a non–decreasing, continuous and concave distortion function, but it is not differentiable at $y = 1 - \alpha$.

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CVaR only accounts for losses in excess of VaR, disregarding losses smaller than the α percentile. It does not adequately account for extreme losses that have a low frequency either, as it is based on an average loss.

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Hence, CVaR leads also to inconsistent decisions.

Example 4.1:

A and B are 2 portfolios with the following loss distributions:

Loss	Distribution A	Distribution B
0	0.600	0.600
1	0.375	0.390
5	0.025	—
11	—	0.010

Table 4.1: From Wang (2002, AFIR Coll.)

Example 4.1 (...continued):

The distorted distributions of these losses can be obtained.

Loss	$S_A(x)$	$S_A^*(x)$
$X < 0$	1	1
$0 \leq X < 1$	0.4	1
$1 \leq X < 5$	0.025	0.5
$5 \leq X$	0	0

Table 4.2: Distorted Distribution for Portfolio A

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Then the $CVaR_\alpha(X_A)$, at level $\alpha = 0.95$, is given by:

$$CVaR_{0.95}(X_A) = \int_0^1 dx + \int_1^5 0.5dx = 3.$$

Example 4.1 (...continued):

Similarly for Portfolio B we obtain:

$$CVaR_{0.95}(X_B) = \int_0^1 dx + \int_1^{11} 0.2dx = 3 = CVaR_{0.95}(X_A),$$

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Similarly for Portfolio B we obtain:

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Wang (2002, AFIR Coll.) suggests that the problem is with the distortion function for CVaR, that assigns 0 to all percentiles below the level of confidence α.

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Similar problems can be observed in cases where only $g(0) = 0$, but the distortion risk measure still leads to inconsistencies.

Example 4.2:

Consider the following distortion function:

$$g_1(x) = \begin{cases} 50x & \text{if } 0 \leq x < 0.01 \\ 0.5 & \text{if } 0.01 \leq x < 0.5 \\ x & \text{if } 0.5 < x \leq 1 \end{cases} .$$

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It is continuous, but not differentiable at $x = 0.01$ and $x = 0.5$, yet constant on $[0.01, 0.5]$.

Example 4.2 (. . . continued):

Here Portfolios A and B have the same maximal loss of 11, but differ on the smaller losses:

Loss	Distribution A	Distribution B
0	0.600	0.600
1	—	0.390
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Table 4.3: Modified Example of Wang (2002, AFIR Coll.)

Again here, the distortion risk measure generated by g_1 assigns the same value (that is 5.5) to both portfolios. Yet, it is clear that Portfolio A is riskier.

Example 4.2 (. . . continued):

The problem is not the non–differentiability of g_1, as suggested in Wang (2002, AFIR Coll.), but the fact that it is constant on an interval, resulting in information loss on the original distribution (for values between 1 and 10).

Loss	Distribution A	Distorted A
0	0.600	0.5
10	0.375	0
11	0.025	0.5

Table 4.4: Distorted Distribution for Portfolio A

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Loss	Distribution A	Distorted A
0	0.600	0.5
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Table 4.4: Distorted Distribution for Portfolio A

At this stage, is natural to turn to the study of the convexity or concavity properties of distortion functions.

Example 4.2 (. . . continued):

Consider another distortion function:

$$g_2(x) = \begin{cases} \frac{1}{3}x & \text{if } 0 \leq x < \frac{1}{3} \\ \frac{4}{3}x - \frac{1}{3} & \text{if } \frac{1}{3} \leq x \leq 1 \end{cases}. \tag{3}$$

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With it Portfolio A is correctly evaluated as riskier (≈ 2) than Portfolio B ($\approx 0,23$).

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This idea defines a new set of complete risk measures.

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Definition 4.1: Consider a risk X and a distortion measure m, i.e. $m(X) = \mathbb{E}_{P^*}(X)$. The risk measure m is called complete if:

$$S(x_1) = S(x_2) \Leftrightarrow S^*(x_1) = S^*(x_2), \quad \forall\, x_1, x_2 \in [0, 1], \quad (4)$$

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where S^* is the survival function of the distorted distribution P^*.

VaR and CVaR are not complete. This shows how their inconsistencies are due to their constant distortion functions over some intervals.

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1. $S(x_1) = S(x_2) \;\Leftrightarrow\; S^*(x_1) = S^*(x_2)$, for all $x_1, x_2 \;\in [0, 1]$,
2. g is a strictly increasing distortion function.

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The difference between convex and concave distortion functions is that they generate super–additive or sub–additive risk measures, respectively.

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Here we are interested in sub–additive risk measures (equivalent to coherence for distortion risk measures). This leads to the idea of exhaustive risk measures.

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The difference between convex and concave distortion functions is that they generate super–additive or sub–additive risk measures, respectively.

Hence sub–additivity (super–additive) is not sufficient. In addition risk measures need to be complete.

Here we are interested in sub–additive risk measures (equivalent to coherence for distortion risk measures). This leads to the idea of exhaustive risk measures.

Definition 4.2: A distortion risk measure is called exhaustive if it is coherent and complete.

Exhaustive (...continued):

From Theorem 4.1 is can be seen that a distortion risk measure is exhaustive if and only if its distortion function, g, is concave and strictly increasing.

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It follows that a simple sufficient condition to check exhaustivity is that g be:

(i) concave and,

(ii) $g(1-) < 1$.

3.3 Beyond exhaustive risk measures

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The answer in no, as illustrated by the following example.

Example 5.1:

Consider the following distortion function:

$$g_3(y) = \begin{cases} \frac{3}{2}y & \text{if} \quad 0 \le y < \frac{1}{2} \\ \frac{1}{2}y + \frac{1}{2} & \text{if} \quad \frac{1}{2} \le y \le 1 \end{cases}, \tag{5}$$

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applied to these two portfolios:

Loss	Distribution A	Distribution B
0	0.500	0.4500
1.9	—	0.350
2	0.375	—
x	—	0.200
8	0.125	—

Table 5.1: Loss Distributions of Portfolios A and B

Example 5.1 (...continued):

It is easily seen that g_3 is concave and satisfies (4), hence it is also exhaustive.

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Nevertheless, a choice of $x \approx 5.74$ in the above portfolios forces g_3 to assign identical risk measure values,

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\begin{aligned}
m(X_A) &= \int_0^2 (\frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2})\, dx + \int_2^8 \frac{3}{2} \cdot \frac{1}{8}\, dx = \frac{21}{8}, \\
m(X_B) &= \int_0^{1.9} (\frac{1}{2} \cdot 0.55 + \frac{1}{2})\, dx + \int_{1.9}^{x} \frac{3}{2} \cdot 0.2\, dx = \frac{21}{8},
\end{aligned}
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to both, even though Portfolio B is clearly more risky.

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to both, even though Portfolio B is clearly more risky.

So it seems that exhaustivity is not a sufficient condition either to avoid inconsistent decisions.

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